LINEAR THERMODYNAMIC THEORY OF HEAT CONDUCTION WITH MEMORY

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Restrictions on the relaxation functions of heat conduction, which are necessary and sufficient for the fulfillment of the thermodynamic postulates in a linear theory of heat conduction with memory, are obtained.

1. A renewal of interest in the last two decades in the thermodynamics of complex materials has led to a clearer realization that there are insufficient grounds for assuming that the usual formulation of the second law of thermodynamics in the form of the Clausius–Duhem inequality is the only one possible. Thus, a series of studies has appeared in which various generalizations and modifications of the formulation of the second law are considered [1-7]. In this connection, it is important to be able to apply to actual materials those consequences of thermodynamic theory which would be necessary and sufficient for the fulfillment of all of its postulates. Comparison with experiment would allow one to judge the completeness of the thermodynamic approach being used. Of particular interest here are model materials with memory because, due to their diverse properties, it is possible to more completely see the consequences of a thermodynamic theory in these cases [3, 8-10].

In the present paper, in the framework of thermodynamic theories based on the Clausius–Duhem inequality, we obtain the necessary and sufficient conditions of thermodynamic admissibility for a linear model of heat conduction with memory, i.e., the complete set of restrictions due to the Clausius–Duhem inequality for this case. It is shown that these restrictions do not prohibit wave solutions of the heat-conduction equation with growing amplitudes. Evidently this indicates that the formulation of the second law with the Clausius–Duhem inequality is not complete.

An attempt to prove a necessary and sufficient restriction on the relaxation functions for a linear theory of heat conduction with memory was initiated by us earlier in [11]. However, the result obtained there cannot be considered to be completely satisfactory because the sufficiency of the obtained restriction could be proved only when an additional condition was applied to the relaxation functions. In any case, this result cannot be considered to be the final answer on the search for a complete set of thermodynamic restrictions. In addition, it turned out that this additional condition led to serious mathematical difficulties.

Below we prove the required necessary and sufficient condition for the fulfillment of the second law of thermodynamics for a linear model of heat conduction with memory. This can be done because we have removed the requirement that the thermodynamic potential be smooth in Hilbert space with fading memory from the starting point formulation of the thermodynamic theory and replaced this requirement by a weaker assumption. The second law of thermodynamics is formulated on the basis of the Clausius–Duhem inequality

$$\eta \geqslant -\operatorname{div}\left(\frac{\vec{q}}{\theta}\right) + \frac{r}{\theta}. \tag{1}$$

For the purposes of studying the properties of the relaxation function, it is sufficient to consider a linear theory of heat conduction; thus, as independent variables we can use the inverse temperature

$$\vartheta = 1/\theta \tag{2}$$

and its gradient

$$\vec{G} = \nabla\vartheta, \tag{3}$$

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since, according to [12], only in this case will the linear thermodynamic theory of heat conduction be correct.

2. We now formulate the starting point of the theory. The thermodynamic potential Φ is introduced by the relation

$$\Phi = \eta - e\vartheta. \tag{4}$$

Then, with the help of the equation of conservation of energy

$$\dot{e} = -\operatorname{div}\vec{q} + r \tag{5}$$

and (2)-(4), the Clausius–Duhem inequality (1) can be transformed to

$$\dot{\Phi} - e\dot{\vartheta} - \vec{q}\cdot\vec{G} \leqslant 0. \tag{6}$$

For simplicity we consider a one-dimensional, linear theory of heat conduction with memory given by the following defining equations:

$$\begin{gathered} e(t) = \hat{e}(\vartheta(t), \vartheta^{(t)}) \overset{\text{def}}{=} e_0 - c\vartheta(t) - \int_0^\infty \beta(s)\vartheta^t(s)\,ds, \\ q(t) = \hat{q}(G^t) = \int_0^\infty \alpha(s) G^t(s)\,ds, \end{gathered} \tag{7}$$

where $\vartheta^t(s) \overset{\text{def}}{=} \vartheta(t-s)$ is the inverse temperature history, and $G^t(s) \overset{\text{def}}{=} G(t-s)$ is the gradient of the inverse temperature history. These equations are obtained by linearization (in the isotropic case) of the modified [12] general Gurtin–Pipkin model [13] of heat conduction with a finite heat propagation velocity.

The relaxation functions α and β are bounded and have the following properties:

$$\int_0^\infty |\beta(s)|\, s^2 ds < \infty, \quad \int_0^\infty |\alpha(s)|\, ds < \infty, \tag{8}$$

$$\alpha(\infty) = 0, \; \beta(\infty) = 0. \tag{9}$$

Let F be the set of piecewise-continuous functions on $(-\infty, \infty)$, bounded on any finite interval, with a propagator bounded to the left.

We define an admissible process (at point x) as the pair $\{\vartheta(t), G(t)\}$, in which $\dot{\vartheta} \overset{\text{def}}{=} \dfrac{d\vartheta}{dt} \in F$, $G \in F$ and $\vartheta > 0$. This means that for any admissible process there exists a t_0 such that $\vartheta = \vartheta_0$ and $G = 0$ for all $t < t_0$. The combination of propagators $\dot{\vartheta}$ and G is called the propagator of the admissible process. The admissible process with the empty propagator is called the equilibrium admissible process.

Each admissible process (for fixed t) defines a triplet consisting of a number and two functions on $[0, \infty)$ $\{\vartheta(t), \vartheta^t(\cdot), G^t(\cdot)\}$, which we call the thermal history. The equilibrium thermal history is of the form $\{\vartheta_0, \vartheta_0^+, 0^+\}$, where $\vartheta_0^+(s) = \vartheta_0, 0^+(s) = 0$ for all $s \in [0, \infty)$.

A thermodynamic process is defined as the set of four functions $\{\vartheta(t), G(t), e(t), q(t)\}$, where $\{\vartheta(t), G(t)\}$ define the admissible process and e(t) and q(t) are defined through this admissible process with the help of (7).*

Here we use a thermodynamic postulate which differs from the postulates of the modified Gurtin–Pipkin theory in that, instead of assuming the smoothness of the thermodynamic potential in a space with fading memory, we use a weaker assumption, and the Clausius–Duhem inequality (6), since Φ is not assumed to be differentiable, is written in integral form.

Postulate TD. There exists a thermodynamic potential Φ, defined on the set of thermal histories by the functional Φ:

$$\Phi(t) = \hat{\Phi}(\vartheta(t), \vartheta^t, G^t), \tag{10}$$

*Any thermodynamic process is consistent with the conservation of energy equation (5) for an appropriate choice of the internal heat source.

which a) for all thermodynamic processes and any $t_1, t_2 \in (-\infty, \infty)$ satisfies the Clausius–Duhem inequality

$$\hat{\Phi}(\vartheta(t_2), \vartheta^{t_2}, G^{t_2}) - \hat{\Phi}(\vartheta(t_1), \vartheta^{t_1}, G^{t_1}) \leqslant \int_{t_1}^{t_2} [\hat{e}(\vartheta(\tau), \vartheta^{\tau})\dot{\vartheta}(\tau) + \hat{q}(G^{\tau})G(\tau)]\, d\tau, \tag{11}$$

and b) is a minimum in the equilibrium state

$$\hat{\Phi}(\vartheta(t), \vartheta^{t}, G^{t}) \geqslant \hat{\Phi}(\vartheta(t), \vartheta(t)^{+}, 0^{+}) \overset{\text{def}}{=} \Phi^{*}(\vartheta(t)) \tag{12}$$

for any thermal history $\{\vartheta(t), \vartheta^{t}, G^{t}\}$.

The function $\Phi^{*}(\vartheta)$, defined in (12), is called the equilibrium thermodynamic potential.

We emphasize that condition (12) on the minimum for the equilibrium thermodynamic potential is weaker than the requirement that it be smooth in the space of fading memory because when there is fading memory, (12) turns out to be a consequence of the Clausius–Duhem inequality (see [13], relations (4) and (8) therein).

3. We now study the properties of the relaxation functions β and α derived from the restrictions imposed by postulate TD. We take an arbitrary admissible process and consider inequality (12) from t_0 (where t_0 is the point on the time axis for which the propagator of the process lies to the right) to arbitrary $t > t_0$:

$$\hat{\Phi}(\vartheta(t), \vartheta^{t}, G^{t}) - \Phi^{*}(\vartheta_0) \leqslant \int_{-\infty}^{t} [\hat{e}(\vartheta(\tau), \vartheta^{\tau})\dot{\vartheta}(\tau) + \hat{q}(G^{\tau})G(\tau)]\, d\tau. \tag{13}$$

The lower limit of integration is extended to $-\infty$ because $\dot{\vartheta} = G = 0$ for $t < t_0$.

Using the properties of the thermodynamic potential given by (12), inequality (13) is transformed to

$$\Phi^{*}(\vartheta(t)) - \Phi^{*}(\vartheta_0) \leqslant \int_{-\infty}^{t} [e(\vartheta(\tau), \vartheta^{\tau})\, \dot{\vartheta}(\tau) + \hat{q}(G^{\tau})\, G(\tau)]\, d\tau. \tag{14}$$

If during the admissible process under consideration the inverse temperature remains constant and equal to ϑ_0, and only its gradient changes, then from (14), the second of equations (7), and a change of variables we obtain

$$\int_{-\infty}^{t} \int_{-\infty}^{\tau} \alpha(\tau - s)\, G(s)\, G(\tau)\, ds\, d\tau \geqslant 0 \tag{15}$$

for any $G \in F$ and any t.

This inequality expresses a property of the relaxation function α imposed by postulate TD. Following [14], we call a relaxation function having property (15) dissipative.*

Below we consider an admissible process in which $G(\tau) = 0$ for all τ and $\vartheta(\tau)$ is given by the following special form:

$$\vartheta(\tau) = \begin{cases} \vartheta_0 & \text{for } \tau \in (-\infty, t_0), \\ \vartheta_0 + \nu \int_{t_0}^{\tau} f(s)\, ds & \text{for } \tau \in [t_0, t], \\ \vartheta_0 + \nu I_f & \text{for } \tau \in [t, t + \lambda), \\ \vartheta_0 + \nu \left[1 - \frac{1}{T}(\tau - t - \lambda)\right] I_f & \text{for } \tau \in [t + \lambda, t + \lambda + T), \\ \vartheta_0 & \text{for } \tau \in [t + \lambda + t, \infty). \end{cases} \tag{16}$$

Here $f \in F$, $\lambda > 0$, $T > 0$, $I_f = \int_{t_0}^{t} f(s)\, ds$, and the factor $\nu > 0$ is chosen such that $\vartheta(\tau) > 0$ for all τ. From the definition of $\vartheta(\tau)$ in (16), it is clear that $\{\vartheta(\tau), 0^{+}\}$ is an admissible pro-

*In mathematical papers a function satisfying (15) is called a function of positive type.

cess. We write (14) with $t = t_1$ for this process. From (7), and the boundedness from the left of the propagator of f, we can transform this inequality to the following form by a substitution of variables, integration by parts, and division by ν^2:

$$\int_{-\infty}^{t}\int_{-\infty}^{\tau}\beta^i(\tau-s)f(s)f(\tau)\,ds\,d\tau+\frac{I_f}{T}\int_{t}^{t+\lambda+T}\int_{-\infty}^{t}\beta^i(\tau-s)f(s)\,ds\,d\tau+\left(\frac{I_f}{T}\right)^2\int_{t+\lambda}^{t+\lambda+T}\int_{t+2}^{\tau}\beta^i(\tau-s)\,ds\,d\tau\geqslant 0, \quad (17)$$

where the integral relaxation function of the internal energy is

$$\beta^i(s)\overset{\text{def}}{=}\int_{s}^{\infty}\beta(\lambda)\,d\lambda \quad (18)$$

and from the first of equations (8), satisfies

$$\int_{0}^{\infty}|\beta^i(s)|\,s\,ds<\infty. \quad (19)$$

By changing variables and the order of integration we transform the last two integrals in (17) and obtain an upper estimate to the second integral:

$$0\leqslant\int_{-\infty}^{t}\int_{-\infty}^{\tau}\beta^i(\tau-s)f(s)f(\tau)\,ds\,d\tau+\frac{I_f}{T}\int_{0}^{T}\int_{-\infty}^{t}\beta^i(t+\lambda+\tau-s)f(s)\,ds\,d\tau+$$
$$+\left(\frac{I_f}{T}\right)^2\int_{0}^{T}\beta^i(s)(T-s)\,ds\leqslant\int_{-\infty}^{t}\int_{-\infty}^{\tau}\beta^i(\tau-s)f(s)f(\tau)\,ds\,d\tau+|I_f|(t-t_0)\mu_{\beta^i}(\lambda)\mu_f-\left(\frac{I_f}{T}\right)^2\int_{0}^{T}\beta^i(s)\,s\,ds+\frac{I_f^2}{T}\int_{0}^{T}\beta^i(s)\,ds. \quad (20)$$

Here t_0 is a point which bounds the propagator of f from the left, $\mu_{\beta^i}(\lambda)$ is the maximum value of the function β^i on the interval $[\lambda, \infty]$, and μ_b is the maximum value of f on the segment $[t_0, t]$. Because the memory functions are bounded, the two maximum values are finite and

$$\lim_{\lambda\to\infty}\mu_{\beta^i}(\lambda)=0, \quad (21)$$

This follows by definition.

Taking the limit $\lambda \to \infty$ in (20), we see that, according to (21), the second term after the last inequality sign goes to zero.

If we put $T \to \infty$, then the two last terms in (20) also go to zero since the first of equations (8) is satisfied. After taking these two limits in the above order, we obtain from (20)

$$\int_{-\infty}^{t}\int_{-\infty}^{\tau}\beta^i(\tau-s)f(s)f(\tau)\,ds\,d\tau\geqslant 0 \quad (22)$$

for any $f\in F$ and any t.

Summarizing the results (15), (22), we conclude that, in order to satisfy postulate TD, the relaxation functions α and β^i must be dissipative.

4. It can be shown that the requirement that the relaxation functions α and β^i be dissipative is not only a necessary but also a sufficient condition for the fulfillment of the thermodynamic postulate in a linear theory. In order to prove this, we construct a functional $\tilde{\Phi}(\vartheta(t),\vartheta^t,G^t)$ starting from the dissipative inequalities (15) and (22) and defined for all admissible processes and satisfying all of the requirements of postulate TD. Using some ideas of Day [3], which were developed for determining the entropy functional for a viscoelastic, non-heat-conducting medium, we construct the required functional.

Let the relaxation functions α and β^i be dissipative so that

$$\int_{-\infty}^{t'}\int_{-\infty}^{\tau}\beta^i(\tau-s)f(s)f(\tau)\,ds\,d\tau\geqslant 0,\qquad\int_{-\infty}^{t'}\int_{-\infty}^{\tau}\alpha(\tau-s)g(s)g(\tau)\,ds\,d\tau\geqslant 0 \quad (23)$$

for any $f, g\in F$.

Let F_t be the set of functions of F whose propagator is bounded and lies on the interval $[t, \infty)$. For any admissible process $\{\vartheta(\cdot), G(\cdot)\}$, any t, and any pair of functions $\eta, h \in F_t$ it is possible to construct a pair of functions $f_{\vartheta_\eta}, g_{G_h} \in F$ as follows:

$$f_{\vartheta_\eta}(\tau) = \begin{cases} \vartheta(\tau) & \text{for } \tau \in (-\infty, t), \\ \eta(\tau) & \text{for } \tau \in (t, \infty), \end{cases}$$
$$g_{G_h}(\tau) = \begin{cases} G(\tau) & \text{for } \tau \in (-\infty, t), \\ h(\tau) & \text{for } \tau \in (t, \infty). \end{cases} \tag{24}$$

Inequality (23) can be written for these functions and, using the fact that both functions have bounded propagators, the inequalities can be represented in the form

$$\int_t^\infty \eta(\tau)\left[\int_0^\infty \beta^i(s) f_{\vartheta_\eta}(\tau - s)\, ds\right] d\tau \geqslant -\int_{-\infty}^t \int_{-\infty}^\tau \beta^i(\tau - s)\,\dot{\vartheta}(\tau)\vartheta(s)\, ds d\tau,$$
$$\int_t^\infty h(\tau)\left[\int_0^\infty \alpha(s) g_{G_h}(\tau - s)\, ds\right] d\tau \geqslant -\int_{-\infty}^t \int_{-\infty}^\tau \alpha(\tau - s) G(\tau) G(s)\, ds d\tau. \tag{25}$$

Now for constant $\vartheta(\cdot)$ and $G(\bullet)$, let the functions $\eta(\bullet)$, $h(\bullet)$ range over the entire set F. The right-hand sides of both inequalities in (25) do not change and, consequently, the expressions on the left of these inequalities are bounded below for any $\eta, h \in F_t$. But this means they also have greatest lower bounds which we will call H_e and H_q, respectively:

$$\int_t^\infty \eta(\tau)\left[\int_0^\infty \beta^i(s) f_{\vartheta_\eta}(\tau - s)\, ds\right] d\tau \geqslant H_e \geqslant -\int_{-\infty}^t \int_{-\infty}^\tau \beta^i(\tau - s)\,\dot{\vartheta}(\tau)\,\dot{\vartheta}(s)\, ds d\tau,$$
$$\int_t^\infty h(\tau)\left[\int_0^\infty \alpha(s) g_{G_h}(\tau - s)\, ds\right] d\tau \geqslant H_q \geqslant -\int_{-\infty}^t \int_{-\infty}^\tau \alpha(\tau - s) G(\tau) G(s)\, ds d\tau \tag{26}$$

for any $\eta, h \in F_t$.

H_e is a functional depending on $\vartheta^t - \vartheta(t)^+$, while H_q is a functional depending on G^t, as can easily be seen from the following equations, obtained after simple manipulations of the expressions on the left-hand side of (26):

$$\hat{H}_e(\vartheta^t - \vartheta(t)^+) \overset{\text{def}}{=} \inf_{\eta \in F_t} \left\{\int_t^\infty \eta(t)\left[\int_t^\tau \eta(s)\beta^i(\tau - s)\, ds + \int_0^\infty (\vartheta^t(s) - \vartheta(t))\,\beta(s + \tau - t)\, ds\right] d\tau\right\},$$

$$\hat{H}_q(G^t) \overset{\text{def}}{=} \inf_{h \in F_t} \left\{\int_t^\infty h(t)\left[\int_t^\tau h(s)\alpha(\tau - s)\, ds + \int_0^\infty G^t(s)\alpha(s + \tau - t)\, ds\right] d\tau\right\}. \tag{27}$$

Putting $h(\tau) \equiv \eta(\tau) \equiv 0$ in (26), we find that functionals $\hat{H}_e$ and $\hat{H}_q$ satisfy

$$\hat{H}_e(\vartheta^t - \vartheta(t)^+) \leqslant 0, \quad \hat{H}_q(G^t) \leqslant 0. \tag{28}$$

In addition, it follows from (26) and (28) that

$$\hat{H}_e(0^+) = 0, \; \hat{H}_q(0^+) = 0. \tag{29}$$

We write an inequality of type (26) for the same functions f_{ϑ_η} and g_{G_h}, but for any time $t' < t$. Splitting the integrals on the left into two, we obtain

$$\int_{t'}^t \dot{\vartheta}(\tau)\int_0^\infty \beta^i(s)\,\dot{\vartheta}(\tau - s)\, ds d\tau + \int_t^\infty \eta(\tau)\int_\tau^\infty f_{\vartheta_\eta}(\tau - s)\, ds d\tau \geqslant \hat{H}_e(\vartheta^{t'} - \vartheta(t')^+),$$
$$\int_{t'}^t G(\tau)\int_0^\infty \alpha(s) G(\tau - s)\, ds d\tau + \int_t^\infty h(\tau)\int_0^\infty g_{G_h}(\tau - s)\, ds d\tau \geqslant \hat{H}_q(G^{t'}). \tag{30}$$

If now both functions h and η range over the entire set F_t, then in (29) only the second integrals are changed and the inequalities are always satisfied. This means that they are satisfied when the integrals are replaced by their greatest lower bounds given by (27). Hence,

$$\int_{t'}^{t} \dot{\vartheta}(\tau) \int_0^\infty \beta^i(s)\, \dot{\vartheta}(\tau - s)\, ds d\tau + \hat{H}_e(\vartheta^t - \vartheta(t)^+) \geqslant \hat{H}_e(\vartheta^{t'} - \vartheta(t')^+),$$
$$\int_{t'}^{t} G(\tau) \int_0^\infty \alpha(s)\, G(\tau - s)\, ds d\tau + \hat{H}_q(G^t) \geqslant \hat{H}_q(G^{t'}). \tag{31}$$

We introduce the notation

$$\hat{\Phi}(\vartheta(\tau),\ \vartheta^t,\ G^t) \overset{\text{def}}{=} e_0\vartheta(t) - \frac{1}{2}(c + \beta^i(0))\, \vartheta(t)^2 - \hat{H}_e(\vartheta^t - \vartheta(t)^+) - \hat{H}_q(G^t). \tag{32}$$

Then combining the two inequalities (31), taking into account (7) and (32), we obtain

$$\int_{t'}^{\tau} [\hat{e}(\vartheta(\tau),\ \vartheta^\tau)\dot{\vartheta}(\tau) + \hat{q}(G^\tau)\, G(\tau)]\, d\tau \geqslant \hat{\Phi}(\vartheta(t),\ \vartheta^t,\ G^t) - \hat{\Phi}(\vartheta(t'),\ \vartheta^{t'},\ G^{t'}). \tag{33}$$

And this is the Clausius–Duhem inequality in which the functional $\hat{\Phi}$ defined by relations (32) and (27) plays the role of a thermodynamic potential. From (32) and (29) we can find the equilibrium thermodynamic potential

$$\Phi^*(\vartheta) = \Phi(\vartheta,\ \vartheta^+,\ 0^+) = e_0\vartheta - \frac{c + \beta^i(0)}{2}\, \vartheta^2. \tag{34}$$

Then from (28) and (32) and (34) it follows from postulate TD that the functional Φ constructed above be a minimum in the equilibrium state. Hence, this functional, together with the defining equations (7), satisfy all of the requirements of postulate TD, and this means that the dissipative property of the relaxation functions β^i and α on which the construction of $\hat{\Phi}$ is based, is sufficient for the fulfillment of this postulate.

The results of Secs. 2 and 3 can be summarized in the form of a theorem.

THEOREM 1. The thermodynamic postulate TD is satisfied for a medium defined by (7) if and only if the integral relaxation function for the internal energy β^i and relaxation function for the heat flux α are dissipative.

We note that, according to [11, 14], the requirement that the relaxation functions be dissipative is equivalent to the requirement

$$0 \leqslant \overline{\beta}^i_c(\omega) \overset{\text{def}}{=} \int_0^\infty \beta^i(s) \cos\omega s ds,\quad 0 \leqslant \overline{\alpha}_c(\omega) \overset{\text{def}}{=} \int_0^\infty \alpha(s) \cos\omega s ds \tag{35}$$

for any $\omega \geqslant 0$. Therefore, the following lemma to Theorem 1 is true.

LEMMA. Condition (35) is necessary and sufficient to satisfy postulate TD.

5. The above theorem contains a complete set of thermodynamic restrictions from the Clausius–Duhem inequality in the case of the heat-conduction model with memory considered here; therefore, one can study whether these restrictions completely exclude unphysical situations. In order to do this, we substitute (7) into the energy equation (5) for $r = 0$ and obtain a linear integrodifferential heat-conduction equation

$$c\dot{\vartheta} + \int_0^\infty \beta(s)\, \dot{\vartheta}(x,\ t - s)\, ds = \int_0^\infty \alpha(s) \frac{\partial^2}{\partial x^2} \vartheta(x,\ t - s)\, ds. \tag{36}$$

We look for a solution to this equation in the form of damped plane harmonic waves with frequency ω and the following dispersion relation for the wave velocity v and damping factor ξ results:

$$v^2(\omega) = \frac{2\omega\, |\overline{\alpha}(\omega)|}{|c + \overline{\beta}(\omega)|\, (1 - \sin(\nu(\omega) - \varphi(\omega)))}, \tag{37}$$

$$\xi(\omega) = \frac{\omega}{v(\omega)}\,\frac{1+\sin(\nu(\omega)-\varphi(\omega))}{\cos(\nu(\omega)-\varphi(\omega))}, \tag{38}$$

where $\bar{\alpha}(\omega) = \bar{\alpha}_c(\omega) - i\bar{\alpha}_c(\omega)$; $\bar{\beta}(\omega) = \bar{\beta}_c(\omega) - i\bar{\beta}_c(\omega)$; $\nu(\omega) = \arg\bar{\alpha}(\omega)$; $\varphi(\omega) = \arg(c+\bar{\beta}(\omega))$; and $\bar{\alpha}_s$ and $\bar{\beta}_s$ are the Fourier sine formations of α and β, in analogy with (35).

We consider the sign of the damping factor (38). It is known that in a viscoelastic medium the thermodynamic restrictions on the stress relaxation function analogous to (35) require that the damping factor of acoustic waves be positive. The analogous exclusion of temperature waves with growing amplitude in the case considered here follows from (35) for $\beta = 0$. However, in the general case ($\beta \neq 0$), this result cannot be proven, and the counterexample discussed below shows that the proof is impossible. Actually, the sign of the damping factor ξ, as is clear from (38), is determined by $\cos(\nu-\varphi)$ which, using standard trigonometric transformations and the properties of the Fourier transform, can be represented as follows:

$$\cos(\nu(\omega)-\varphi(\omega)) = [(c+\beta^i(0)-\omega\bar{\beta}_s^i(\omega))\,\bar{\alpha}_c(\omega) + \omega\bar{\beta}_c^i(\omega)\,\bar{\alpha}_s(\omega)]\,\frac{1}{|\bar{\alpha}(\omega)|\;|c+\bar{\beta}(\omega)|}. \tag{39}$$

We consider now functions α and β^i of the following form

$$\beta^i = \tau\beta_0 e^{-s/\tau},$$
$$\alpha(s) = \frac{\alpha_0 T^2}{s^2}\left[\sin^2\left(\frac{s}{T}\right) + \sin^2\left(\frac{2s}{T}\right) - 2\sin^2\left(\frac{3s}{2T}\right)\right], \tag{40}$$

where α_0 and β_0 are positive constants and $\tau > 0$ and $T > 0$ are certain characteristic times.

The Fourier cosine transforms of these functions will have the forms [15]:

$$\bar{\beta}_c^i(\omega) = \frac{\beta_0\tau^2}{(1+\tau^2\omega^2)},$$
$$\bar{\alpha}_c(\omega) = \begin{cases} 0 & \text{for } \omega\in\left[0,\frac{2}{T}\right) \text{ and } \omega\in\left(\frac{4}{T},\infty\right), \\ \frac{\pi}{4}\alpha_0 T^2\left(\omega-\frac{2}{T}\right) & \text{for } \omega\in\left[\frac{2}{T},\frac{3}{T}\right], \\ \frac{\pi}{4}\alpha_0 T^2\left(\frac{4}{T}-\omega\right) & \text{for } \omega\in\left[\frac{3}{T},\frac{4}{T}\right]. \end{cases} \tag{41}$$

It is easily seen that both of these functions are nonnegative for any $\omega \geqslant 0$ and, according to the lemma to Theorem 1, this means that the relaxation functions (40) are thermodynamically admissible. But the Fourier sine transforms of the relaxation functions (40) are [15]

$$\bar{\beta}_s^i(\omega) = \frac{\beta\omega\tau^3}{(1+\omega^2\tau^2)},$$
$$\bar{\alpha}_s(\omega) = \frac{\alpha_0 T^2}{4}\left[\left(\omega+\frac{2}{T}\right)\ln\left|\omega+\frac{2}{T}\right| + \left(\omega-\frac{2}{T}\right)\ln\left|\omega-\frac{2}{T}\right| + \right. \tag{42}$$
$$\left. + \left(\omega+\frac{4}{T}\right)\ln\left|\omega+\frac{4}{T}\right| + \left(\omega-\frac{4}{T}\right)\ln\left|\omega-\frac{4}{T}\right| - \left(\omega+\frac{3}{T}\right)\ln\left|\omega+\frac{3}{T}\right| - \left(\omega-\frac{3}{T}\right)\ln\left|\omega-\frac{3}{T}\right|\right].$$

With the help of (41) and (42), we calculate (39) for fixed frequency $\omega_0 = 1/T$:

$$\cos\left(\nu\left(\frac{1}{T}\right)-\varphi\left(\frac{1}{T}\right)\right) = \frac{1}{T\left|\bar{\alpha}\left(\frac{1}{T}\right)\right|\left|c+\bar{\beta}\left(\frac{1}{T}\right)\right|}\,\frac{\beta_0\tau^2}{1+\frac{\tau^2}{T^2}}\,\frac{\alpha_0 T}{4}\ln\left(\frac{5^2\cdot 2^4}{2^{16}}\right) < 0. \tag{43}$$

Because the sign of the damping factor (38) is determined by the sign of this cosine, it follows that, at frequency $\omega_0 = 1/T$, temperature waves with increasing amplitude propagate in this case. Such waves are not observed in experiment. Moreover, the existence of special frequencies at which temperature waves grow in time would be unphysical because then equilibrium thermal fluctuations would grow at these frequencies, and this would lead to instability of the equilibrium state. Therefore, the possibility of growing temperature waves must be excluded, in our view. If we accept the natural point of view that growing temperature waves should be excluded by the thermodynamics, then it follows from our results that the Clausius–Duhem inequality does not give a complete set of thermodynamic restrictions.

NOTATION

c, instantaneous volumetric heat capacity; e, internal energy density; G, inverse temperature gradient; q, heat flux; r, power density of the internal heat sources; v, velocity of temperature waves; α and β, relaxation functions for the heat flux and internal energy; β^i, integral relaxation function for the internal energy; η, entropy density; θ, absolute temperature; ϑ, inverse of the absolute temperature; ξ, temperature wave damping factor; Φ, thermodynamic potential.

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